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INTERACTION BETWEEN PARALLEL CRACKS IN LAYERED COMPOSITES†

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Abstract—The plane strain problem of a multi-layered composite with parallel cracks is considered. The main objective of this paper is to study the interaction between parallel and collinear cracks. The problem is formulated in terms of a set of simultaneous singular integral equations which are solved numerically. The effect of material properties on the interaction between cracks is also demonstrated.

INTRODUCTION

Welded and bonded structures have been observed to contain multiple cracks. The study of the interaction between such cracks has also been of considerable interest to reactor designers. The problem of a multi-layered composite containing a single crack was studied by Erdogan and Gupta[1, 2]. The interaction between multiple cracks in an isotropic medium and collinear cracks in a layered composite has been considered by Ratwani[3, 4].

In the present study, the analytical methods of [1–4] have been extended to treat the layered composite containing parallel and collinear cracks. In particular, the plane strain problem of an elastic layer bonded to two dissimilar half-planes is considered. The layer medium contains one or two symmetrically placed collinear flaws and one of the half-planes is assumed to have a single parallel flaw. The procedure, of course, can easily treat any composite containing n elastic layers and cracks located along m parallel planes. For the sake of simplicity, only the symmetric problem is studied here. The anti-symmetric loading case can be handled in an analogous manner.

Stress intensity factors at all the crack tips are computed. Their variation with respect to the crack locations, geometry and material of the composite are presented graphically.

FORMULATION OF THE PROBLEM

Consider the plane problem, shown in Fig. 1, containing one or two collinear cracks in each plane. The cracks are assumed to be located symmetrically with respect to the y-axis. In this paper, our primary interest is in the disturbed stress state caused by the cracks. Hence, assuming that the overall stress distribution $\sigma_{ij}{}^0$ in the imperfection-free medium is known, the stress state $\sigma_{ij}{}^T$ in the cracked medium may be expressed as

$$\sigma_{ij}{}^T = \sigma_{ij}{}^0 + \sigma_{ij} \tag{1}$$

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where σ_{ij} is the disturbed stress state obtained by subjecting the crack surfaces to the following tractions.

$$\begin{aligned}
\sigma_{yy}{}^1(x, h) &= -\sigma_{yy}{}^0(x, h) = p_2(x) \\
\sigma_{xy}{}^1(x, h) &= -\sigma_{xy}{}^0(x, h) = p_1(x), \qquad C < |x| < D, \\
\sigma_{yy}{}^3(x, -h_1) &= -\sigma_{yy}{}^0(x, -h_1) = q_2(x) \\
\sigma_{xy}{}^3(x, -h_1) &= -\sigma_{xy}{}^0(x, -h_1) = q_1(x) \qquad A < |x| < B.
\end{aligned} \tag{2}$$

$p_i(x)$ and $q_i(x)$ satisfy a Hölder condition in their respective ranges.

The integral transform technique, described in detail for a single crack[1] and for multiple cracks[3], is used here to formulate the problem in terms of four unknown functions defined by

$$\begin{aligned}
f_1(x) &= \frac{\partial}{\partial x}(u_2 - u_1), f_2(x) = \frac{\partial}{\partial x}(v_2 - v_1), \qquad (C < |x| < D, \quad y = h), \\
g_1(x) &= \frac{\partial}{\partial x}(u_3 - u_4), g_2(x) = \frac{\partial}{\partial x}(v_3 - v_4), \qquad (A < |x| < B, \quad y = -h_1).
\end{aligned} \tag{3}$$

where: $u_i(x, y)$ and $v_i(x, y)$ are the displacement fields in the respective regions shown in Fig. 1.

Note that the crack surfaces are the singular surfaces across which the displacement vector suffers a discontinuity and the unknown functions define the derivatives of the crack opening displacements. For the sake of simplicity, the central plane of the elastic layer is assumed to have the cracks. The case when the crack lies at the interface between the bielastic media has been treated in detail in [2].

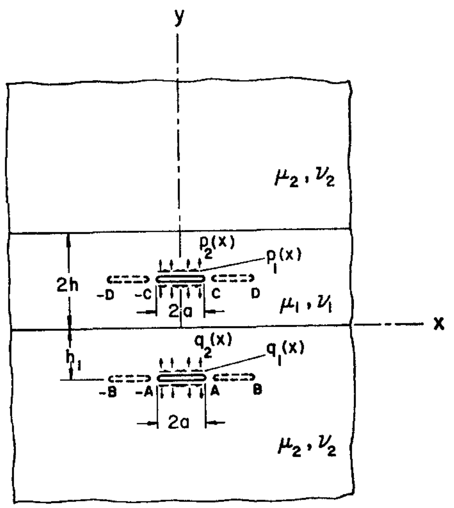

Fig. 1. Geometry of the layered composite with collinear and parallel cracks.

Following the procedure of [1] and [3], a set of simultaneous singular integral equations of the first kind is derived, expressed as follows:

$$\int_{L_1} \sum_{j=1}^{2} f_j(t)\left[\frac{\delta_{ij}}{t-x} + K_{ij}(x,t)\right] \mathrm{d}t + \int_{L_2} \sum_{k=1}^{2} g_k(\tau) H_{ik}(x,\tau)\,\mathrm{d}\tau = \frac{1+\kappa_1}{2\mu_1}\,\pi p_i(x),$$
$$C < |x| < D, \quad i = 1, 2,$$

$$\int_{L_1} \sum_{j=1}^{2} f_j(t) L_{ij}(x,t)\,\mathrm{d}t + \int_{L_2} \sum_{k=1}^{2} g_k(\tau)\left[\frac{\delta_{ik}}{\tau - x} + M_{ik}(x,\tau)\right] \mathrm{d}\tau = \frac{1+\kappa_2}{2\mu_2}\,\pi q_i(x),$$
$$A < |x| < B, \quad i = 1, 2, \qquad (4)$$

where $L_1 \equiv (C < |x| < D)$ and $L_2 \equiv (A < |x| < B)$, and $\kappa_i = 3 - 4\nu_i$ for plane strain and $\kappa_i = (3 - \nu_i)/(1 + \nu_i)$ for generalized plane stress. μ_i and ν_i are the shear modulii and the Poisson's ratios, for $i = 1, 2$, denoting the elastic layer by the subscript 1 and the half-planes by 2. The functions K_{ij}, L_{ij}, H_{ik} and M_{ik} are Fredholm kernels and are bounded in their respective closed intervals. The expressions for these Fredholm kernels are given as follows:

$$K_{11}(x,t) = \int_0^\infty \frac{s_1(\alpha) - 4\alpha h}{D_1(\alpha)} \mathrm{e}^{-2\alpha h} \sin\alpha(t-x)\,\mathrm{d}\alpha$$

$$K_{12}(x,t) = K_{21}(x,t) = 0$$

$$K_{22}(x,t) = \int_0^\infty \frac{s_1(\alpha) + 4\alpha h}{D_2(\alpha)} \mathrm{e}^{-2\alpha h} \sin\alpha(t-x)\,\mathrm{d}\alpha$$

$$H_{11}(x,\tau) = \frac{\mu_2}{\mu_2 - \mu_1}\frac{1+\kappa_1}{2\lambda_3}\int_0^\infty [\lambda_3 s_3(\alpha) + (1 - 1\alpha h_1)s_2(\alpha)]\,.\,\frac{\mathrm{e}^{-\alpha(h+h_1)}}{D_1(\alpha)} \sin\alpha(\tau - x)\,\mathrm{d}\alpha$$

$$H_{12}(x,\tau) = \frac{\mu_2}{\mu_2 - \mu_1}\frac{1+\kappa_1}{2\lambda_3}\int_0^\infty [\lambda_3 s_3(\alpha) - (1 + 2\alpha h_1)s_2(\alpha)]\,.\,\frac{\mathrm{e}^{-\alpha(h+h_1)}}{D_1(\alpha)} \cos\alpha(\tau - x)\,\mathrm{d}\alpha$$

$$H_{21}(x,\tau) = \frac{\mu_2}{\mu_2 - \mu_1}\frac{1+\kappa_1}{2\lambda_3}\int_0^\infty [\lambda_3 s_5(\alpha) - (1 - 2\alpha h_1)s_4(\alpha)]\,.\,\frac{\mathrm{e}^{-\alpha(h+h_1)}}{D_2(\alpha)} \cos\alpha(\tau - x)\,\mathrm{d}\alpha$$

$$H_{22}(x,\tau) = -\frac{\mu_2}{\mu_2 - \mu_1}\frac{1+\kappa_1}{2\lambda_3}\int_0^\infty [\lambda_3 s_5(\alpha) + (1 + 2\alpha h_1)s_4(\alpha)]\,.\,\frac{\mathrm{e}^{-\alpha(h+h_1)}}{D_2(\alpha)} \sin\alpha(\tau - x)\,\mathrm{d}\alpha$$

$$L_{ij}(x,t) = \frac{(\mu_2 - \mu_1)^2(\lambda_1 - \lambda_2)\lambda_3}{\mu_2{}^2(1+\kappa_1)^2} H_{ij}(x,t), \quad i,j = 1,2 \qquad (5)$$

$$M_{11}(x,\tau) = -\frac{1}{2}\int_0^\infty \Bigg[\lambda_3 + s_8(\alpha) + \frac{(\lambda_1 - \lambda_2)}{D_1(\alpha)D_2(\alpha)}\{\lambda_3 s_7(\alpha)$$
$$+ [s_7(\alpha)s_8(\alpha) - 8\alpha h(1 - 2\alpha h_1)]\mathrm{e}^{-4\alpha h}\}\Bigg] \mathrm{e}^{-2\alpha h_1} \sin\alpha(\tau - x)\,\mathrm{d}\alpha$$

$$M_{12}(x,\tau) = -\frac{1}{2}\int_0^\infty \Bigg[\lambda_3 - s_6(\alpha) + \frac{(\lambda_1 - \lambda_2)}{D_1(\alpha)D_2(\alpha)}\{\lambda_3 s_7(\alpha)$$
$$+ [16\alpha^2 h h_1 - s_6(\alpha)s_7(\alpha)]\mathrm{e}^{-4\alpha h}\}\Bigg] \mathrm{e}^{-\alpha h_1} \cos\alpha(\tau - x)\,\mathrm{d}\alpha$$

$$M_{21}(x,\tau) = M_{12}(x,\tau)$$

$$M_{22}(x,\tau) = -\frac{1}{2}\int_0^{\infty}\left[\lambda_3 + s_9(\alpha) + \frac{(\lambda_1-\lambda_2)}{D_1(\alpha)D_2(\alpha)}\{\lambda_3 s_7(\alpha) - [s_7(\alpha)s_9(\alpha) - 8\alpha h(1+2\alpha h_1)]\, e^{-4\alpha h}\}\right] e^{-2\alpha h_1}\sin\alpha(\tau - x)\,d\alpha$$

where

$$\begin{aligned}
D_1(\alpha) &= -\lambda_2 + (4\alpha h + \lambda_1 e^{-2\alpha h})e^{-2\alpha h}\\
D_2(\alpha) &= \lambda_2 + (4\alpha h - \lambda_1 e^{-2\alpha h})e^{-2\alpha h}\\
s_1(\alpha) &= 1 + \lambda_1\lambda_2 + 4\alpha^2h^2 - 2\lambda_1 e^{-2\alpha h}\\
s_2(\alpha) &= -\lambda_2 + (1 + 2\alpha h)e^{-2\alpha h}\\
s_3(\alpha) &= 1 - 2\alpha h - \lambda_1 e^{-2\alpha h}\\
s_4(\alpha) &= \lambda_2 - (1 - 2\alpha h)e^{-2\alpha h}\\
s_5(\alpha) &= -1 - 2\alpha h + \lambda_1 e^{-2\alpha h}\\
s_6(\alpha) &= (1 - 4\alpha^2 h_1^{\,2})/\lambda_3\\
s_7(\alpha) &= -\lambda_2 + \lambda_1 e^{-4\alpha h}\\
s_8(\alpha) &= (1 - 2\alpha h_1)^2/\lambda_3\\
s_9(\alpha) &= (1 + 2\alpha h_1)^2/\lambda_3
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
\lambda_1 &= \frac{\mu_1\kappa_2 - \mu_2\kappa_1}{\mu_2 + \mu_1\kappa_2}\\
\lambda_2 &= \frac{\mu_1 + \mu_2\kappa_1}{\mu_1 - \mu_2}\\
\lambda_3 &= \frac{\mu_2 + \mu_1\kappa_2}{\mu_2 - \mu_1}.
\end{aligned} \tag{7}$$

The unknown functions f_i and g_i in equations (4) have integrable singularities at the end points. Therefore, the equations (4) must be solved subject to the singlevaluedness conditions

$$\int_C^D f_i(t)\,dt = 0 = \int_A^B g_i(t)\,dt, \qquad i = 1, 2. \tag{8}$$

The singular integral equations (4) are solved simultaneously by using the numerical technique described in [5]. It may be noted that if one of the cracks lies on the interface, the corresponding integral equation would become that of second kind. The numerical technique to treat such equations is described in [6] and is used to solve the interface crack problems in [2]. The stress intensity factors $K_{\rm I}$ and $K_{\rm II}$ at all the crack tips are defined as in [1]. As an example, for the crack in medium (1) near the crack tip $x \to D$, these can be expressed as

$$\begin{aligned}
K^1_{\rm I} &= \lim_{x\to D}\sqrt{2(x-D)}\,\sigma_{yy}^{\ 1}(x,h)\\
K^1_{\rm II} &= \lim_{x\to D}\sqrt{2(x-D)}\,\sigma_{xy}^{\ 1}(x,h).
\end{aligned} \tag{9}$$

The stress intensity factors can also be expressed in terms of the unknown functions $f_i(x)$. The functions $f_i(x)$, which have integrable singularities, may be written as

$$f_i(x) = \frac{G_i(x)}{\sqrt{(D-x)(x-C)}}\cdot \tag{10}$$

Equations (9) can now be written as

$$\begin{aligned} K^1{}_{\mathrm{I}} &= -\frac{2\mu_1}{1+\kappa_1}\lim_{x\to D}\sqrt{2(D-x)}f_2(x) \\ &= -\frac{2\mu_1}{1+\kappa_1}\sqrt{\frac{2}{D-C}}G_2(D) \\ K^1{}_{\mathrm{II}} &= \frac{2\mu_1}{1+\kappa_1}\lim_{x\to D}\sqrt{2(D-x)}f_1(x) \\ &= \frac{2\mu_1}{1+\kappa_1}\sqrt{\frac{2}{D-C}}G_1(D). \end{aligned} \tag{11}$$

Superscripts 1 and 2 on the stress intensity factors refer to the cracks in medium 1 and 2 respectively.

DISCUSSION OF RESULTS

To demonstrate the interaction between parallel and collinear cracks, a layered composite as shown in Fig. 1 is assumed to contain parallel cracks, one in the mid-plane of the elastic layer and the other in one of the half planes. A realistic loading condition is that of uniform loads acting far away from the cracks. The problem is usually divided in two parts. First the unflawed layered composite is solved for the actual loading. Normal and shear stress components at the crack locations are computed from this global solution. The second part of the problem is to solve the disturbance problem with the tractions acting on the crack surfaces equal to negative of that obtained in the first (global) solutions. A superposition of the two solutions provides the results for the original problem.

However, the main problem of interest in this paper is the disturbance problem. Two numerical examples are presented. In all cases, an epoxy layer with elastic properties $\mu_1 = 4{\cdot}5 \times 10^5$ psi, $\nu_1 = 0{\cdot}35$ is sandwiched between two aluminum half planes: $\mu_2 = 10^7$ psi and $\nu_2 = 0{\cdot}3$. In the first example, the layered composite is assumed to contain only two parallel cracks. The input tractions were assumed to be uniform uniaxial stresses with zero shear component, i.e. in equation (2)

$$\begin{aligned} p_1(x) &= q_1(x) = 0 \\ p_2(x) &= -\sigma \quad \text{or} \\ q_2(x) &= -\sigma. \end{aligned} \tag{12}$$

Relatively less common, however, equation (12) represents the pressurized crack situation. In a particular problem, one may have arbitrary input tractions with appropriate symmetry conditions. The effect of the distance between the two parallel cracks on the four stress intensity factors (at each crack tip) is presented in Table 1. When the crack in medium 1 is loaded ($p_2(x) = -\sigma$), as expected, we get negative stress intensity factors for crack 2. Similar effect is observed at crack 1, when the crack in medium 2 is loaded ($q_2(x) = -\sigma$). It is clear that the results need to be superimposed if both the cracks are loaded simultaneously.

Table 1. Stress intensity factors vs h_1/a for $h/z = 1{\cdot}0$, $2{\cdot}0$; $p_1(x) = q_1(x) = 0{\cdot}0$; Material 1 = Epoxy ($\mu_1 = 4{\cdot}5 \times 10^5$ psi, $\nu_1 = 0{\cdot}35$); Material 2 = Aluminum ($\mu_2 = 10^7$ psi, $\nu_2 = 0{\cdot}30$); $K_0 = \sigma\sqrt{\pi a}$ (parallel crack configuration)

$h/a = 1{\cdot}0$	$p_2(x) = -\sigma, q_2(x) = 0$				$p_2(x) = 0, q_2(x) = -\sigma$			
h_1/a	$K^1{}_{\rm I}/K_0$	$K^1{}_{\rm II}/K_0$	$K^2{}_{\rm I}/K_0$	$K^2{}_{\rm II}/K_0$	$K^1{}_{\rm I}/K_0$	$K^1{}_{\rm II}/K_0$	$K^2{}_{\rm I}/K_0$	$K^2{}_{\rm II}/K_0$
0·2	0·695	0·080	−1·990	−1·033	−0·182	−0·160	3·243	1·251
0·5	0·608	0·020	−1·132	−0·460	−0·064	−0·032	2·102	0·525
1·0	0·580	0·001	−0·620	−0·184	−0·030	−0·011	1·450	0·163
2·0	0·576	0·000	−0·305	−0·078	−0·009	−0·001	1·149	0·031
∞	0·582	0·0	0·0	0·0	0·0	0·0	1·0	0·0
$h/a = 2{\cdot}0$								
0·2	0·817	0·027	−1·030	−0·494	−0·108	−0·040	2·738	0·978
0·5	0·785	0·008	−0·677	−0·251	−0·042	−0·015	1·997	0·473
1·0	0·770	0·000	−0·393	−0·102	−0·020	−0·008	1·430	0·157
2·0	0.770	0·000	−0·220	−0·053	−0·009	−0·000	1·146	0·031
∞	0·775	0·0	0·0	0·0	0·0	0·0	1·0	0·0

We observe that the absolute magnitudes of all the K values increase as crack 2 nears the interface. Table 1 also shows an effect of the layered thickness on the results. Notice the decrease in the interaction between the two cracks since they are farther apart now. This effect is explicitly shown in Fig. 2, where crack 1 is loaded only. The stress intensity factors at crack 1 due to this loading increase with increasing layer thickness. In limit when $h \to \infty$, $K^1{}_{\rm I} = 1{\cdot}0$ and $K^1{}_{\rm II} = K^2{}_{\rm I} = K^2{}_{\rm II} = 0$, which is the well known result for a homogeneous material containing a crack. A reverse effect is observed at the stress intensity factors at crack 2 due to the loading at crack 2 itself, as shown in Fig. 3. $K^2{}_{\rm I}$ and $K^2{}_{\rm II}$ decrease as the layer thickness is increased and, in the limiting case when $h \to \infty$, these approach asympotically to the values obtained for the problem of a bimaterial medium containing a single crack in one of the halfplanes. Interaction terms, of course, vanish as $h \to \infty$.

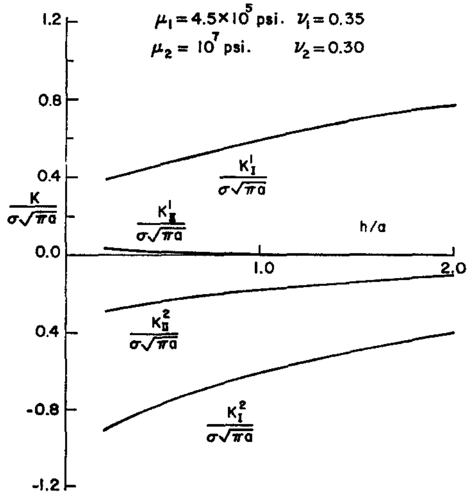

Fig. 2. Stress intensity factors vs the layer thickness h/a for $h_1/a = 1{\cdot}0$; $p_2(x) = -\sigma$, and $p_1(x) = q_1(x) = q_2(x) = 0$; configuration: parallel cracks.

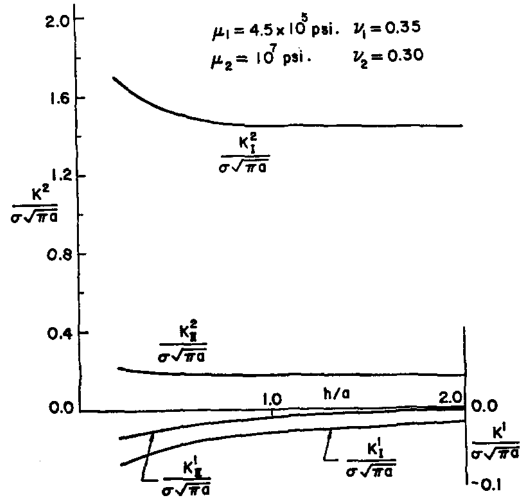

Fig. 3. The case with $q_2(x) = -\sigma$ and $p_1(x) = p_2(x) = q_1(x) = 0$; configuration: same as for Fig. 2.

As a second example, the case of the elastic layer containing two collinear cracks at the mid-plane and located symmetrically is considered. Since the problem is symmetrical, stress intensity factors at only one of the collinear cracks need be computed. In all numerical cases, the location of crack 2 and the layer thickness have been kept fixed. Again, either the collinear cracks or crack 2 is loaded at a time. Variation of the mode I stress intensity factor with respect to the distance between the collinear cracks is shown in Fig. 4. When these

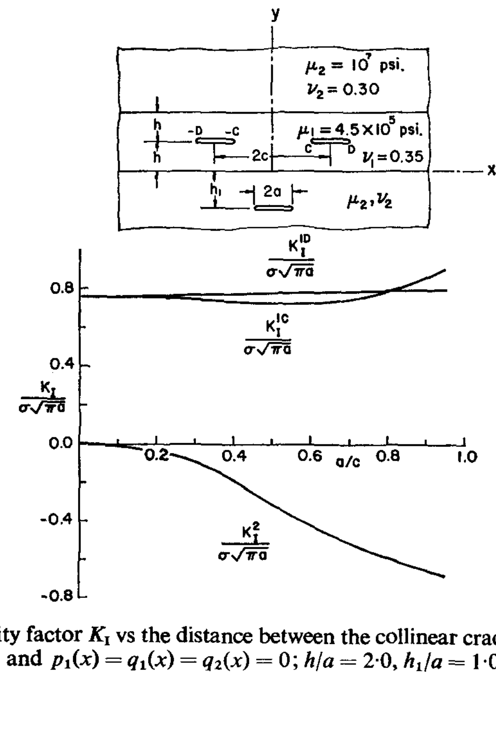

Fig. 4. Stress intensity factor K_I vs the distance between the collinear cracks (a/c); $p_2(x) = -\sigma$, and $p_1(x) = q_1(x) = q_2(x) = 0$; $h/a = 2{\cdot}0$, $h_1/a = 1{\cdot}0$.

collinear cracks are far away, only a little interaction between the parallel crack is observed. Another interesting phenomenon observed is that, if the two collinear cracks are close by, i.e. $c/a > 1{\cdot}25$, the outer crack tips would become unstable before the inner crack tip. When crack 2 is loaded, the stress intensity factors $K^2{}_{\mathrm{I}}$ and $K^2{}_{\mathrm{II}}$ (at the tips of crack 2) remain practically unaffected due to the displacement of the collinear crack locations. However, the interaction between the cracks is quite strongly affected by the distance between the two collinear cracks, as shown in Table 2. As the two cracks come closer, $|K_{\mathrm{I}}|$ at both crack tips increases monotonically, with a faster rise at the inner tip. $|K_{\mathrm{II}}|$ at the inner tip, on the other hand, undergoes a maximum value and steadily decreases to a very low value.

Table 2. Stress intensity factors at the collinear crack tips vs the distance between them (a/c): $q_2(x) = -\sigma$, $p_2(x) = p_1(x) = q_1(x) = 0{\cdot}0$; Configuration: insert in Fig. 4

a/c	Crack tip C		Crack tip D	
	$-\dfrac{K^{1C}{}_{\mathrm{I}}}{\sigma\sqrt{\pi a}}$	$-\dfrac{K^{1C}{}_{\mathrm{II}}}{\sigma\sqrt{\pi a}}$	$-\dfrac{K^{1D}{}_{\mathrm{I}}}{\sigma\sqrt{\pi a}}$	$-\dfrac{K^{1D}{}_{\mathrm{II}}}{\sigma\sqrt{\pi a}}$
0·00	0·00·0	0·0	0·0	0·0
0·20	0·0022	0·0036	0·0004	0·0015
0·35	0·0056	0·0076	0·0018	0·0040
0·50	0·0110	0·0088	0·0048	0·0066
0·65	0·0162	0·0076	0·0072	0·0079
0·80	0·0205	0·0050	0·0092	0·0082
0·95	0·0248	0·0023	0·0106	0·0082

In conclusion, whenever there is a structure containing multiple cracks, an analysis of the type described in this paper can be utilized in order to find the critical configurations under which the structure may be most vulnerable. The result, in short, shows that cracks situated at various locations in a structure do interact with each other and this interaction becomes very important when they are rather closely spaced. In such cases, the strength predictions would be much more adequate and safe if these interactions have been taken into account.

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Абстракт—Рассматривается проблема плоской деформации многослойного составного вещества с параллельными трещинами. Главной целью этой работы является изучение взаимодействия между параллельными и коллинеарными трещинами. Проблема формулируется в терминах ряда совместных сингулярных интегральных уравнений, которые разрешаются численно. Также демонстрируется эффект свойств материала на взаимодействие между трещинами.